The New Keynesian Model with Stochastically Varying Policies

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Abstract

The Multiplicative Ergodic Theorem provides a novel general methodology to analyze rational expectations models with stochastically varying coefficients. The approach is applied for the first time to economics and analyzes the canonical New Keynesian model with a Taylor rule which switches randomly between an aggressive and a passive reaction to inflation. The paper delineates the trade-off of the central bank of being passive in some periods and aggressive in others. Moreover, it is shown how this trade-off depends on the stochastic process governing the randomness in the central bank's policy. Finally, explicit solution formulas are derived in the case of determinateness as well as indeterminateness. In doing so he paper considerably extends the current approach.

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Keywords: time–varying rational expectations models, New Keynesian model, Taylor rule, Lyapunov exponents, multiplicative ergodic theorem

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1 Introduction

The presupposition of constant coefficients in affine (linear) rational expectations macroeconomic models is a very tenuous position. Indeed there are several convincing reasons to believe in time-varying coefficients instead. First, time-varying coefficient models arise naturally from the linearization of nonlinear models along solution paths (Elaydi, 2005, p. 219–220). Second, the relationships describing the economy undergo structural changes giving rise to drifting coefficients as emphasized by Lucas' critique. F.e. Sargent (1999) provides an interpretation in terms of self-confirming equilibria. Third, policies and policy rules are subject to change. Cogley and Sargent (2005), Primiceri (2005), or Chen, Leeper, and Leith (2015), among many others, provide empirical evidence with regard to U.S. monetary policy.

As convincing as these arguments may be, a rigourous analysis of linear, respectively affine, rational expectations models with stochastically varying coefficients is hindered by the fact that the eigenvalues of the matrices involved provide in general no information about the stability of the underlying difference equation.[1] Hence the spectral theorem (see Meyer, 2000, chapter 7.2) which underlies the usual solution formulas in the case of constant coefficients is no longer applicable. Fortunately, there is perfect substitute for the eigenvalues in terms of *Lyapunov exponents*. The Lyapunov exponents are defined as the asymptotic growth rates of the endogenous variables along solution paths. The celebrated Oseledets Multiplicative Ergodic Theorem (MET) which is at the heart of the theory of *random dynamical systems* then lifts the eigenvalue\eigenvector analysis typically used in the constant coefficient case to the case of stochastically varying coefficients using the Lyapunov exponents\spaces.[2] Thus, the MET provides a perfect substitute for the spectral theorem and, consequently, will allow the derivation of explicit solution formulas for rational expectations model with stochastically varying coefficients. These solution formulas turn out to be in the spirit of Blanchard and Kahn (1980), Klein (2000), and Sims (2001) and are therefore directly interpretable in economic terms. There is, however, a price to pay for

[1]This is a well-known fact. Elaydi (2005, p. 191), Colonius and Kliemann (2014, pp. 109–110), and Neusser (2017, appendix A) present several simple examples to illustrate this claim. Francq and Zakoïan (2001) provide further illustrations in a time series context.

[2]Colonius and Kliemann (2014) provides a clear and accessible presentation of the MET by relating it to the standard eigenvalue\eigenspace analysis. The monograph by Arnold (2003) and Viana (2014) also provide elaborated and excellent expositions, but are mathematically more evolved. Neusser (2017) presents a first application of the MET to an economic model.

this generalization: the Lyapunov exponents and their associated Lyapunov spaces cannot, in general, be computed analytically, but are only accessible via numerical procedures. This alleged disadvantage is compensated by powerful numerical algorithms which do not only allow the computation of the Lyapunov exponents, but also their corresponding Lyapunov spaces (see Dieci and Elia (2008), Froyland et al. (2013), and Neusser (2017) for details). The contribution of this paper is the presentation of a comprehensive theory for analyzing and solving random coefficient rational expectations models. The theory relies on the powerful MET which is applied for the first time to analyze dynamic models in economics.

This paper shares the ambition of Farmer, Waggoner, and Zha (2009) and Farmer, Waggoner, and Zha (2011) to provide a solid and adequate solution methodology for forward-looking Markov-switching rational expectations models. These papers rely on the mean square stability concept as proposed by Costa, Fragoso, and Marques (2005) in the context of optimal control theory and effectively focus on the size of the spectral radius of a specific matrix (see Farmer, Waggoner, and Zha (2009, equation (14)) or Davig and Leeper (2007, proposition 1), but also Bougerol and Picard (1992) and Francq and Zakoïan (2001, theorem 2) in the context of Markov-switching multivariate ARMA models, and Foerster et al. (2016, section 4.2) and Barthélemy and Marx (2017) in the context of nonlinear models). This approach is mathematically equivalent to the analysis of the top (largest) Lyapunov exponent and is sufficient, at least for the examples considered, to characterize the stability of the model. However, this exclusive focus on the top Lyapunov exponent disregards the rich information encoded in the Lyapunov spectrum (set of all Lyapunov exponents) and the associated Lyapunov spaces. This becomes particularly evident when characterizing the properties of more sophisticated models with some initial conditions and\or indeterminate models (see Proposition 2 of this paper).

We use the canonical New Keynesian macroeconomic model with Taylor rule as a vehicle to expose these new concepts and to demonstrate their practicability and usefulness. This application is, however, not just an illustrative example, but provides results which are of interest in their own right. It is well-known that, in this model, the central bank must respond aggressively against inflation in order to obtain a unique solution (determinateness). When the central bank is passive, the model fails to have a unique solution (indeterminateness). An extreme situation arises when the policy is based on central bank 's projection taking the interest rate path as given. This implies that the Taylor rule is effectively eliminated from the model which then becomes indeterminate.[3] According to Galí (2011) this is or has been the practice at many central banks. He goes on to discuss remedies of the resulting indeterminacy problem. In particular, he discusses the possibility that a Taylor rule with an aggressive central bank is restored at some *known fixed* date in the future (see also Laséen and Svensson, 2011, for a similar analysis). More in line with the scope of this paper, Davig and Leeper (2007) investigate the consequences of a regime-switching Taylor rule. Their analysis, however, relies on a restrictive and perhaps inadequate setting as argued by Farmer, Waggoner, and Zha (2010). This paper proposes a more comprehensive analysis of the issue of randomly switching monetary policy rules. Thereby we delineate the trade-off of the central bank of being passive in some periods and aggressive in others. Moreover, it is shown how this trade-off depends on the stochastic process governing the randomness in the central bank's policy. Finally, we provide an explicit solution formula for the determinate as well as for the indeterminate case.

The paper proceeds by first reviewing the New Keynesian model with constant coefficients. This allows to introduce the notation and to connect to the standard literature. We then analyze the random coefficient case theoretically. Based on this analysis, we present some simulation results by considering random switches between an active and a passive monetary policy against inflation. A conclusion finally closes the paper.

2 The New Keynesian Model

2.1 The Setup

The canonical New Keynesian macroeconomic model is one workhorse of modern macroeconomics and has therefore been extensively analyzed in the literature. In this paper we investigate the determinacy of this model and take the microeconomic foundation as given. The papers most closely related to this one are Lubik and Schorfheide (2004), Farmer, Waggoner, and Zha (2009), and Galí (2011). The model typically comprises the following three equations:

$$
\begin{aligned}
y_t &= \mathbb{E}_t y_{t+1} - \sigma^{-1}(i_t - \mathbb{E}_t \pi_{t+1}) + u_t^d, && \text{(IS-equation)} \\
\pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa y_t + u_t^s, && \text{(forward-looking Phillips-curve)} \\
i_t &= \phi_t^\pi \pi_t + \phi_t^y y_t, && \text{(Taylor-rule)}
\end{aligned}
$$

[3]For an assessment of the Taylor rule see the papers collected in Koenig, Leeson, and Kahn (2012).

where the endogenous variables y_t, π_t, and i_t denote income (output gap), the rate of inflation and the nominal interest rate. u_t^d and u_t^s are exogenous demand and supply shocks, respectively. All variables are indexed by time $t \in \mathbb{Z}$. The structural parameters of the IS-equation and the Phillips-curve are supposed to be fixed and to obey the following restrictions: $\sigma > 0$, $\kappa > 0$, and $0 < \beta < 1$. In contrast, the parameters of the Taylor-rule ϕ_t^π and ϕ_t^y are considered to vary *randomly* over time according to an exogenously given regular Markov chain which will be specified in detail in Sections 2.3 and 3. The coefficients of the Taylor rule are assumed to satisfy $\phi_t^\pi \geq 0$ and $\phi_t^y \geq 0$, independently of t. If there is no confusion, the time index is sometimes omitted for simplicity. Finally, $\mathbb{E}_t$ denotes, as usual, the conditional expectations operator based on information up to and including period t (see Appendix A for details). This information includes, in particular, the knowledge of the mechanism generating the randomness of the coefficients.

For technical reasons we have to place some restriction on the stochastic process $\{u_t\} = \{(u_t^d, u_t^s)'\}$. A very weak assumption is the following integrability condition.

Assumption 1 (Integrability)**.**

$$\mathbb{E} \log^+ \|(u_t^d, u_t^s)'\| < \infty$$

where $\log^+ x = \max\{0, \log x\}$.

This assumption is weaker than $\mathbb{E} \log \|(u_t^d, u_t^s)'\| < \infty$. Note that $\{(u_t^d, u_t^s)'\}$ is allowed to be autocorrelated. Hence, u_t can be specified as an autoregressive process which is often assumed in practice.

The model can be expressed in terms of $x_{t+1} = (y_{t+1}, \pi_{t+1})'$ by inserting the Taylor-rule in the IS-equation to obtain the following affine random coefficient expectational difference equation:

$$\mathbb{E}_t x_{t+1} = G F_t x_t - G u_t = A_t x_t + b_t, \quad t \in \mathbb{Z}, \tag{2.1}$$

where

$$G = \begin{pmatrix} 1 & -1/(\beta\sigma) \\ 0 & 1/\beta \end{pmatrix} \quad \text{and} \quad F_t = \begin{pmatrix} 1 + \phi_t^y/\sigma & \phi_t^\pi/\sigma \\ -\kappa & 1 \end{pmatrix}.$$

Thus, the New Keynesian model has the format of a *boundary value problem.* It consists of the affine expectational difference equation (2.1) and a *boundedness constraint*:

Constraint (boundedness constraint)**.** *There exists* $M \in \mathbb{R}$ *such that*

$$\|x_t\| < M < \infty \text{ for all } t \in \mathbb{Z}$$

where $\|.\|$ *is a suitable norm.*

The above constraint is supposed to hold almost surely. Note that the New Keynesian model as outlined above has no initial conditions.[4] The model will be called determinate if the boundary value problem (i.e. the difference equation (2.1) plus the boundedness constraint) has a unique solution. Otherwise the model is said to be indeterminate.

If $\{x_t^{(1)}\}$ and $\{x_t^{(2)}\}$ are two solutions of the difference equation (2.1), then $\{x_t^{(1)} - x_t^{(2)}\}$ satisfies the linear expectational difference equation

$$\mathbb{E}_t x_{t+1} = A_t x_t. \tag{2.2}$$

This implies that the superposition principle holds and that every solution $\{x_t\}$ of the affine difference equation (2.1) is of the form

$$x_t = x_t^{(g)} + x_t^{(p)}$$

where $x_t^{(g)}$ denotes the general solution of the linear equation (2.2) and $x_t^{(p)}$ a particular solution to the general equation (2.1).

In order to find the general solution to the linear equation, define the random matrix product $\{\Phi(t)\}$ as

$$\Phi(t) = \begin{cases} A_{t-1} \dots A_1 A_0, & t = 1, 2, \dots; \\ I_2, & t = 0; \\ A_t^{-1} \dots A_{-1}^{-1}, & t = -1, -2, \dots \end{cases}$$

Note that $\Phi(t)$ is well-defined because the parameter restrictions of the model imply that A_t is nonsingular, i.e. $A_t \in \mathbb{GL}(2)$, for all $t \in \mathbb{Z}$, irrespective of the values of ϕ_t^π and ϕ_t^y.[5] When we want to emphasize the dependence on the realization of the stochastic process, we write $A(\theta^t \omega)$ for A_t and $\Phi(t, \omega)$ for $\Phi(t)$ where $\omega \in \Omega$ is an outcome from the underlying probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ and where θ denotes the time shift operator (see Appendix A for details).

Next define a new variable m_t as $m_t = \Phi(t)^{-1} x_t$. It is easy to see that $\{m_t\}$ is a martingale:

$$\mathbb{E}_t m_{t+1} = \mathbb{E}_t \left(\Phi(t+1)^{-1} x_{t+1} \right) = \Phi(t+1)^{-1} \mathbb{E}_t x_{t+1} = \Phi(t+1)^{-1} A_t x_t = m_t.$$

Similarly, the time reversed process $\tilde{m}_t = m_{-t}$, $t \in \mathbb{Z}$, is also a martingale. This implies without any additional assumptions that there exists a random variable x such that $\lim_{t \to \infty} m_t = x$ a.s. and in mean (see Grimmett and

[4]The approach can be easily generalized to encompass models with initial conditions (see Neusser, 2017)

[5]The determinant of A_t is $1/\beta + (\kappa \phi_t^\pi + \phi_t^y)/\beta\sigma > 1$ (see Section 2.2).

Stirzaker, 2001, section 12.7). Moreover, the original martingale can be reconstructed from x by setting $m_t = \mathbb{E}(x \mid \mathfrak{F}_t)$. Thus, the space of martingales can be continuously parameterized by the space of random variables which are measurable with respect to $\mathfrak{F} = \sigma\left(\bigcup_{t\in\mathbb{Z}} \mathfrak{F}_t\right)$.[6] This implies that the general solution of the linear equation (2.2) can be represented as

$$x_t = (A_{t-1} \dots A_1 A_0)\, x_0 = \Phi(t)x.$$

where x is some random variable measurable with respect to $\mathfrak{F}$. Given some realization $\omega \in \Omega$, the solutions (orbits) are then denoted by $x_t = \varphi(t, \omega, x) = \Phi(t, \omega)x(\omega)$.

The existence and the stability properties of the solutions given by equation (2.1) thus depend crucially on the convergence of the matrix product $\Phi(t, \omega)$. To study this issue, we introduce the notion of *Lyapunov exponents* $\lambda(\omega, x)$. These exponents are defined as the asymptotic growth rates of solutions of the linear random dynamical system $x_{t+1} = A_t x_t = A(\theta^t\omega)x_t$ taking $x_0 = x \neq 0$ as a starting value:

$$\lambda(\omega, x) = \limsup_{t\to\infty} \frac{1}{t} \log \|\varphi(t, \omega, x)\|. \tag{2.3}$$

In the case of a constant coefficient matrix, $\varphi(t, \omega, x) = A^t x$ and the Lyapunov exponents are just the logarithms of the distinct moduli $|\mu_k|$ of the eigenvalues μ_k of A.[7] In the case of random coefficients, the Multiplicative Ergodic Theorem (see Arnold (2003), Colonius and Kliemann (2014), or Viana (2014)) implies under some general technical assumptions (see Appendix A for details) that there exists, in our case, two real numbers, called Lyapunov exponents, λ_{max} and λ_{min} (often called extremal Lyapunov exponents) with $\infty > \lambda_{max} \geq \lambda_{min} > -\infty$.[8] These exponents will be constants independent of $\omega \in \Omega$ and $x \in \mathbb{R}^2$ and will be approached as limits:

$$\lambda_{max} = \lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t, \omega)x\| \geq \lambda_{min} = \lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t, \omega)^{-1}x\|^{-1}.$$

The Appendix A provides further details and shows alternative characterizations of the Lyapunov exponents. Moreover, it is shown there how the Lyapunov exponents and the associated Lyapunov spaces serve as a substitute for eigenvalues and eigenspaces in the standard constant coefficient case.

[6]Compare this to Klein (2000, Definition 4.3 and Assumption 4.2)

[7]See Colonius and Kliemann (2014, section 1.5).

[8]In the case of n-dimensional systems there may $\ell \leq n$ distinct Lyapunov exponents.

2.2 Constant Coefficients

Although the eigenvalues of the "time frozen" coefficient matrices are uninformative with respect to the stability of the system, it is nevertheless instructive to investigate the constant coefficient case in detail (see the references in footnote 1). Denote for this purpose by A the coefficient matrix where ϕ_t^π and ϕ_t^y take specific values ϕ^π and ϕ^y which remain constant over time. The characteristic polynomial of A, $\mathcal{P}(\mu)$, with corresponding eigenvalues μ_1 and μ_2, is then given by

$$\mathcal{P}(\mu) = (\mu - \mu_1)(\mu - \mu_2) = \mu^2 - \mathrm{tr}(A)\mu + \det A$$

with

$$\begin{aligned}
\mathrm{tr}A = \mu_1 + \mu_2 &= 1 + \frac{1}{\beta} + \frac{\kappa}{\sigma\beta} + \sigma^{-1}\phi^y > 2,\\
\det A = \mu_1\mu_2 &= \frac{1}{\beta} + \frac{\kappa\phi^\pi + \phi^y}{\sigma\beta} > 1,\\
\Delta = (\mathrm{tr}A)^2 - 4\det A &= \left(1 - \frac{1}{\beta}\right)^2 + \frac{\kappa}{\sigma\beta}\left(\frac{\kappa}{\sigma\beta} + 2 + \frac{2}{\beta} - 4\phi^\pi\right)\\
&\quad + \frac{\phi^y}{\beta\sigma}\left(\beta\sigma^{-1}\phi^y + 2\beta + 2\kappa\sigma^{-1} - 2\right),\\
\mathcal{P}(1) = (1-\mu_1)(1-\mu_2) &= \frac{\kappa}{\sigma\beta}(\phi^\pi - 1) + \sigma^{-1}(\beta^{-1} - 1)\phi^y
\end{aligned}$$

where Δ denotes the discriminant of the quadratic equation $\mathcal{P}(\mu) = 0$. Note that, irrespective of the parameters, A is nonsingular because $\det A > 1$. Depending on ϕ^π, the roots of $\mathcal{P}(\mu)$ may be complex. We therefore distinguish two cases:

(i) ϕ^π is so large such that $\Delta < 0$. In this case we have two complex conjugate roots. Assuming that $\kappa\sigma^{-1} > 1 - \beta$, a very plausible assumption, this case can only arise if $\phi^\pi > 1$. Because $\det A > 1$, they are both located outside the unit circle.[9] The model is determinate and the unique solution compatible with the boundedness constraint is one where the initial value x_0 is equal to zero so that $x_t = x_t^{(p)}$.

(ii) ϕ^π is small enough such that $\Delta > 0$. In this case there are two distinct real eigenvalues. They must also be of the same sign because the determinant of A is positive. From $\mathrm{tr}\, A > 2$, we infer that they must both be

[9] Another way to reach this conclusion is by observing that the real part of the roots is $\frac{\mathrm{tr}A}{2} > 1$.

positive and that at least one eigenvalue is bigger than one. From the expression of $\mathcal{P}(1)$, we finally conclude that both eigenvalues are bigger than one if and only if $\phi^\pi > 1-\frac{\phi^y}{\kappa}(1-\beta)$. A sufficient condition for this is that $\phi^\pi > 1$. If this condition holds the model is determinate and the only initial condition compatible with the boundedness constraint is again $x_0 = 0$ and $x_t = x_t^{(p)}$. If the central bank is passive with respect to inflation, i.e. if $\phi^\pi < 1 - \frac{\phi^y}{\kappa}(1-\beta)$, the model is indeterminate.

The results of this discussion are summarized in the bifurcation diagram drawn in Figure 1 which plots the Lyapunov exponents as a function of ϕ^π for alternative values of ϕ^y.[10] Consider first the (standard) case where the central bank does not react to output (blue line), i.e. where $\phi^y = 0$. In this situation the stability of the model is independent of the parameters β, κ, and σ and depends solely on the value of ϕ^π. Starting with $\phi^\pi = 0$ and moving progressively to a more aggressive central bank, we first obtain two distinct Lyapunov exponents opposite of zero. Thus, the model is indeterminate. As ϕ^π gradually increases, the distance between the two Lyapunov exponents shrinks. When ϕ^π becomes greater than one, the lower Lyapunov exponent λ_{min} becomes positive and the model determinate. Increasing ϕ^π further, the discriminant Δ becomes negative and the eigenvalues complex conjugate. Hence, the two Lyapunov exponents collapse to a single one. However, the model remains determinate. If the central bank also reacts to output, i.e. if $\phi^y > 0$, the behavior of the model remains qualitatively the same. The differences being that the value of ϕ^π at which the model switches from an indeterminate one to a determinate one is now lower than one and that the value at which the two Lyapunov exponents collapse is larger.[11]

Finally, we derive explicit solution formulas for the boundary value problem consisting of the expectational difference equation (2.1) and the boundedness constraint. To do so, we make the following additional assumption.

Assumption 2 (Hyperbolicity: Constant Coefficients)**.** *A has no eigenvalue on the unit circle. A matrix with this property is called hyperbolic.*

From the previous discussion, we deduce that the hyperbolicity of A is equivalent to the assumption $\phi^\pi \neq 1-\frac{\phi^y}{\kappa}(1-\beta)$. Consider first the case where the model is determinate. This is equivalent to the assumption that the moduli of both eigenvalues of A are bigger than one, or equivalently that the Lyapunov exponents are positive. Then, the unique nonexplosive (bounded) solution of the linear equation (2.2) is the zero solution which is obtained by setting

[10] Remember that in the case of a constant coefficient matrix, the Lyapunov exponents are just the logarithms of the distinct moduli $|\mu_k|$ of the eigenvalues μ_k of A.

[11] This second statement assumes $\kappa\sigma^{-1} > 1-\beta$.

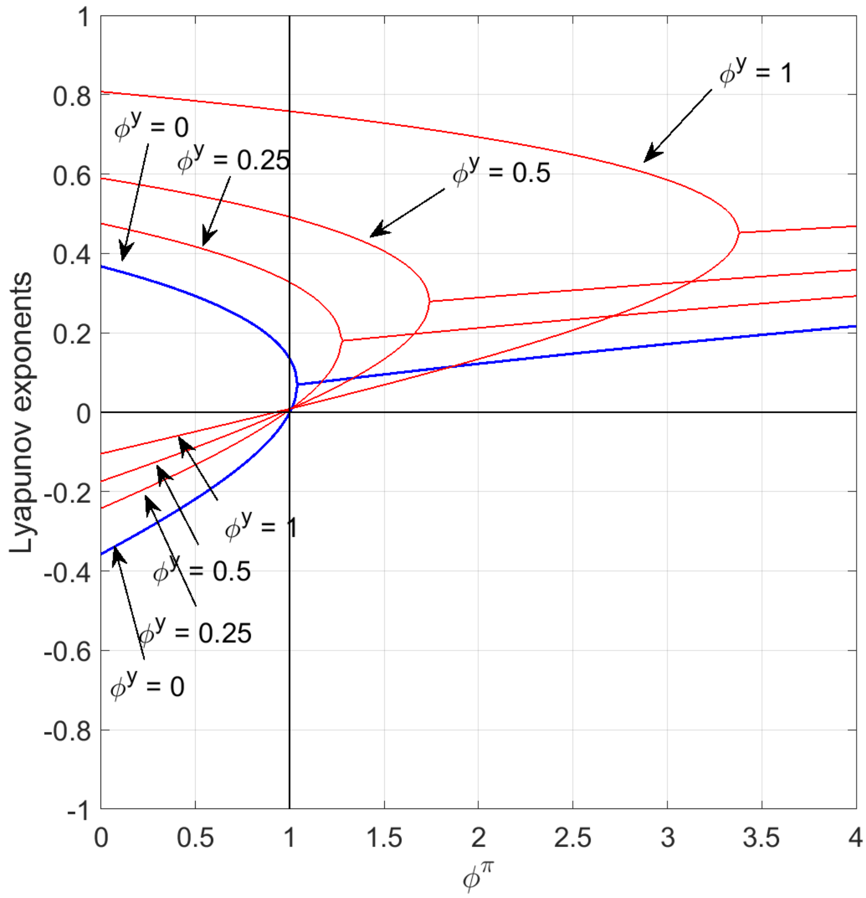

Figure 1: Lyapunov exponents as a function of ϕ^π for different values of ϕ^y ($\beta = 0.99$, $\kappa = 0.132$, $\sigma = 1$)

x_0 equal to zero. A particular solution of the affine difference equation (2.1) then is

$$x_t = x_t^{(p)} = -\sum_{j=1}^{\infty} A^{-j}\mathbb{E}_t b_{t+j-1} = \sum_{j=1}^{\infty} A^{-j} G \mathbb{E}_t u_{t+j-1}. \qquad (2.4)$$

This expression is well-defined if the hyperbolicity assumption 2 holds.

Consider next the case of indeterminacy. In this case there are two positive real and distinct eigenvalues opposite of one. Denote these two eigenvalues by μ^{max} and μ^{min} and their associated eigenvectors by a^{max} and a^{min}. Let $\mathbb{P}^{max}$ and $\mathbb{P}^{min}$ be the projections onto $\mathrm{N}(A - \mu^{max} I_2)$ along $\mathrm{R}(A - \mu^{max} I_2)$ and onto $\mathrm{N}(A - \mu^{min} I_2)$ along $\mathrm{R}(A - \mu^{min} I_2)$, respectively, where N and R denote the nullspace and the column space. These projections can be expressed in terms of matrices as (see Meyer, 2000, chapter 7.2)

$$\begin{aligned}
\mathbb{P}^{max} &= \begin{pmatrix} a^{max} & a^{min} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^{max} & a^{min} \end{pmatrix}^{-1} \\
\mathbb{P}^{min} &= \begin{pmatrix} a^{max} & a^{min} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{max} & a^{min} \end{pmatrix}^{-1}.
\end{aligned}$$

Note that we have $\mathbb{P}^{max} + \mathbb{P}^{min} = I_2$. With this notation, we can represent all bounded solutions in the indeterminate case as

$$\begin{aligned}
x_t &= x_t^{(g)} + x_t^{(p)} \\
&= A^t x_0 + \sum_{j=0}^{\infty} A^j \mathbb{P}^{min} b_{t-j-1} - \sum_{j=1}^{\infty} A^{-j} \mathbb{P}^{max} \mathbb{E}_t b_{t+j-1} \\
&= A^t x_0 - \sum_{j=0}^{\infty} A^j \mathbb{P}^{min} G u_{t-j-1} + \sum_{j=1}^{\infty} A^{-j} \mathbb{P}^{max} (G \mathbb{E}_t u_{t+j-1}) \qquad (2.5)
\end{aligned}$$

with $x_0 \in \mathrm{span}(a^{min})$. Note that x_0 is characterized by $x_0 = \mathbb{P}^{min} x_0$. Because $\mathbb{P}^{max}$ and $\mathbb{P}^{min}$ project on $\mathrm{span}(a^{max})$ and $\mathrm{span}(a^{min})$, respectively, the expression above is well-defined if the hyperbolicity assumption 2 holds. Moreover, we see that the indeterminacy is parameterized by $\mathrm{span}(a^{min})$. The dimension of $\mathrm{span}(a^{min})$ is one because there is always one eigenvalue bigger than one implying $\dim \mathrm{span}(a^{max}) = 1$.

2.3 Random Coefficients

2.3.1 Solution Formulas

We now turn the main contribution of this paper and consider the case where the coefficient matrix A_t is no longer constant, but, due to the randomness of ϕ^{π} and, eventually ϕ^{y}, is varying over time. It is well-known that in such a situation the analysis of the eigenvalues of the "time frozen" coefficient matrices is no longer informative about the stability of the model. It may be the case that the model is unstable although the moduli of the eigenvalues of each A_t considered on its own are less than one (see the references in footnote 1). Fortunately, the Lyapunov exponents defined in equation (2.3) provide a perfect substitute as shown by Oseledets' acclaimed Multiplicative Ergodic Theorem (MET). Appendix A provides a precise statement of the theorem and additional details. One implication of the MET is that, despite the randomness, the Lyapunov exponents are fixed number which are approached as a limit.

In the context of the New Keynesian model the lemma below shows that there is always one positive Lyapunov exponent.

Lemma 1. *The maximal Lyapunov exponent is always strictly greater than zero, i.e.* $\lambda_{max} > 0$.

Proof. Denote by $\delta_{max}(\Phi(t,\omega))$ and $\delta_{min}(\Phi(t,\omega))$ the two singular values of $\Phi(t,\omega)$, then the last assertion of the MET (see appendix) implies

$$\begin{aligned}
\lambda_{max} + \lambda_{min} &= \lim_{t\to\infty} \frac{1}{t} \log \delta_{max}(\Phi(t,\omega)) + \lim_{t\to\infty} \frac{1}{t} \log \delta_{min}(\Phi(t,\omega)) \\
&= \lim_{t\to\infty} \frac{1}{t} \log \left(\delta_{max}(\Phi(t,\omega))\delta_{min}(\Phi(t,\omega))\right) \\
&= \lim_{t\to\infty} \frac{1}{t} \log |\det \Phi(t,\omega)| = \lim_{t\to\infty} \frac{1}{t} \sum_{j=0}^{t} \log |\det A(\theta^j\omega)| \\
&= \mathbb{E} \log |\det A(\omega)|.
\end{aligned}$$

The last equality follows from the Birkoff's ergodic theorem (see, for example, Silva, 2008, theorem 5.1.1). Because $\det A = {}^1\!/_\beta + {}^{(\kappa\phi^\pi + \phi^y)}\!/_{(\sigma\beta)} > {}^1\!/_\beta > 1$, irrespective of the realized values of ϕ^π and ϕ^y, $\lambda_{max} + \lambda_{min} > 0$. Hence, $\lambda_{max} > 0$ as claimed. $\square$

From this lemma we immediately deduce the following two consequences:

(i) The New Keynesian model is *determinate* if and only if both Lyapunov exponents are strictly greater than zero. In this case the only bounded solution of the linear difference equation (2.2) is the zero solution which is obtained by setting $x_0 = 0$.

(ii) The New Keynesian model is *indeterminate* if and only if $\lambda_{min} < 0$. In this case there is an infinite number of initial values $x \in L_{\lambda_{min}}(\omega)$ such that $\varphi(t,\omega,x) = \Phi(t,\omega)x$ converges (exponentially fast) to zero. Hence, $\varphi(t,\omega,x) = \Phi(t,\omega)x$ is a bounded solution satisfying the linear difference equation (2.2). The linear space $L_{\lambda_{min}}(\omega)$ depends on the realization ω and is therefore stochastic, but has constant dimension one. It is called the Lyapunov space associated with λ_{min}.

As in the constant coefficient case, we devise explicit solution formulas for the two cases. This requires, as before, the hyperbolicity of the New Keynesian model viewed as a random dynamical system.

Assumption 3 (Hyperbolicity: Stochastic Case). *$\Phi(t, \omega)$ is hyperbolic, i.e. all Lyapunov exponents are different from zero.*

We are now in a position to deduce from ARNOLD (2003, corollary 5.6.6) and ARNOLD (2003, theorem 5.6.5) directly the solution formula for each case.

Proposition 1 (Solution: Determinateness). *Under the assumptions of the MET, the integrability condition 1 for u_t, and the hyperbolicity assumption 3, the New Keynesian model is determinate if and only if $\lambda_{min} > 0$. The unique invariant solution is*

$$\begin{aligned} x_t &= -\Phi(t)\ \mathbb{E}_t\left[\sum_{j=1}^{\infty} \Phi(t+j)^{-1} b_{t+j-1}\right] \\ &= \ \ \Phi(t)\ \mathbb{E}_t\left[\sum_{j=1}^{\infty} \Phi(t+j)^{-1} (G\,u_{t+j-1})\right]. \end{aligned} \tag{2.6}$$

Proposition 2 (Solution: Indeterminateness). *Under the assumptions of the MET, the integrability condition 1 for u_t, and the hyperbolicity assumption 3, the New Keynesian model is indeterminate if and only if $\lambda_{min} < 0$. The set of invariant solutions is given by*

$$\begin{aligned} x_t = \Phi(t)x_0 + \Phi(t)\ &\sum_{j=0}^{\infty} \Phi(t-j)^{-1}\, \mathbb{P}^{min}_{t-j}\, b_{t-j-1} \\ &- \Phi(t)\ \mathbb{E}_t\left[\sum_{j=1}^{\infty} \Phi(t+j)^{-1}\, \mathbb{P}^{max}_{t+j}\, b_{t+j-1}\right] \\ = \Phi(t)x_0 - \Phi(t)\ &\sum_{j=0}^{\infty} \Phi(t-j)^{-1}\, \mathbb{P}^{min}_{t-j}\, Gu_{t-j-1} \\ &+ \Phi(t)\ \mathbb{E}_t\left[\sum_{j=1}^{\infty} \Phi(t+j)^{-1}\, \mathbb{P}^{max}_{t+j}\, Gu_{t+j-1}\right] \end{aligned} \tag{2.7}$$

where $x_0(\omega) \in L_{\lambda_{min}}(\omega)$. $\mathbb{P}^{max}_{t+j}$ and $\mathbb{P}^{min}_{t-j}$ are the projections onto $L_{\lambda_{max}}(\theta^{t+j}\omega)$ along $L_{\lambda_{min}}(\theta^{t+j}\omega)$, respectively onto $L_{\lambda_{min}}(\theta^{t-j}\omega)$ along $L_{\lambda_{max}}(\theta^{t-j}\omega)$.

The major difference to the constant coefficient case is that the Lyapunov spaces $L_{max}(\omega)$ and $L_{min}(\omega)$ which serve as a substitute for the eigenspaces are time-varying and dependent on the realization of the stochastic process governing the randomness of A. This implies the corresponding projections have also to be random. This stands in contrast to the Lyapunov exponents which are fixed numbers.

The solution formulas above clearly show the attractiveness of the approach based on the MET. The Lyapunov spectrum (the set of Lyapunov exponents) and the associated Lyapunov spaces encode all the knowledge necessary for the understanding of the dynamic properties of the model. As shown by the two propositions above, they reveal much more information than just about the stability of the system. Compare this to FRANCQ and ZAKOÏAN (2001), DAVIG and LEEPER (2007), or FARMER, WAGGONER, and ZHA (2009) who investigate the stability of their model by analyzing the spectral radius of a certain matrix. They effectively focus only on the top Lyapunov exponent (the largest Lyapunov exponent). Moreover, the solution formulas above make sense intuitively and conform with the standard constant coefficient case. Hence the technique exemplified here represents a natural extension of the standard procedures outlined in BLANCHARD and KAHN (1980), KLEIN (2000), and SIMS (2001).

2.3.2 Specification of Randomness

In order to apply these results, we have to be concrete and specify the stochastic process governing the randomness of A_t in detail. In particular, we assume that A_t is drawn randomly from a finite set $\{A(\phi_i^\pi) \mid i = 1, 2, \dots, n)\}$ where $A(\phi_i^\pi)$ denotes the matrix A with value $\phi^\pi = \phi_i^\pi$. ϕ^y is assumed to be constant across states. Furthermore, the randomness is governed by a regular (irreducible (ergodic) and aperiodic) stationary Markov chain with n states and transition probabilities $(P)_{ij}$, $i, j = 1, 2, \dots, n$. For simplicity, we assume $n = 2$ so that the transition matrix P can be written as

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}.$$

Thus $(P)_{ij} = \mathbf{P}[A_{t+1} = A(\phi_j^\pi) \mid A_t = A(\phi_i^\pi)]$. If $p, q \in (0, 1)$, the chain is regular with invariant distribution $\delta = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$. Thus, δ is the unique distribution which satisfies $\delta P = \delta$. Hence, the chain is on average in $q/(p+q)$ percent of the time in state one and $p/(p+q)$ percent of the time in state two. Although the dependency of the Lyapunov exponents on the underlying stochastic process is a subtle issue and still unresolved in general, MALHEIRO and VIANA (2015) have shown that in the specification considered here the Lyapunov exponents depend continuously on the coefficients of the transition matrix. This makes the following simulation exercise a meaningful undertaking.

Noting that the mean exit time from state i is $1/(1-(P)_{ii})$, we measure the mobility of the chain by the mobility index $M(P)$ which has been proposed by SHORROCKS (1978):

$$M(P) = \frac{n - \mathrm{tr}P}{n-1}.$$

$M(P)$ is nothing but the reciprocal of the harmonic mean of the mean exit times.[12] In the specification above $M(P) = p + q$. Thus, the amount of mobility is maximized if the chain switches deterministically (i.e. if $p = q = 1$) between the two states[13] and minimized if the chain stays in its initial state (i.e. if $p = q = 0$).

3 Simulation Results

There is a large number of dimensions along which the model can be simulated. In the following we report those of which we hope will be the most interesting ones for the reader. We consider the specification $\phi_1^\pi = 0$ versus $\phi_2^\pi > 0$ with $\phi^y = 0$ in both states:

$$A(0) = \begin{pmatrix} 1 + \kappa/\beta\sigma & -1/\beta\sigma \\ -\kappa/\beta & 1/\beta \end{pmatrix}, \qquad A(\phi_2^\pi) = \begin{pmatrix} 1 + \kappa/\beta\sigma & (\phi^\pi\beta - 1)/\beta\sigma \\ -\kappa/\beta & 1/\beta \end{pmatrix}.$$

In state one where $\phi_1^\pi = 0$ the central does not react to inflation at all. When this is the case, the nominal interest rate becomes exogenous. This situation arises when central banks base their policy on an explicit inflation forecast taking the interest rate path as given. According to GALÍ (2011) this is or has been a common practice in many central banks. This specification results in an indeterminate model in the constant coefficient case. See GALÍ (2011) for an economic interpretation and possible remedies. In particular, Gallí discusses the possibility of switching back to an inflation sensitive central bank policy after some *given and fixed* horizon (see also LASÉEN and SVENSSON, 2011, for a similar analysis).

[12] SHORROCKS (1978) provides an axiomatic foundation for this index.

[13] The case of periodically switching coefficients results in models which can be analyzed in the context of Floquet's theory. See ELAYDI (2005, section 3.4) and COLONIUS and KLIEMANN (2014, section 7.1) for excellent expositions. An application of this theory to the model under scrutiny here is provided in NEUSSER (2017).

In state two $A_t = A(\phi_2^\pi)$ with $\phi_2^\pi > 0$. Hence, there is a response of the central bank to inflation. This response must be larger than one to obtain a determinate model in the constant coefficient case. In both states, there is no feedback by the central bank to output, i.e. $\phi^y = 0$. The remaining parameters are $\kappa = 0.132$, $\sigma = 1$, and $\beta = 0.99$ which correspond to those in FARMER, WAGGONER, and ZHA (2009).

Although the Lyapunov exponents play a similar role as the eigenvalues do in the constant coefficient case, they cannot, in general, be found analytically. Instead, they can be approximated numerically by simulations. The challenge is that $\varphi(t, \omega, x) = \Phi(t, \omega)x$ tends to align in the direction of fastest growth very quickly leading to a numerical overflow on any computer. To avoid this difficulty, we use the product QR algorithm as discussed in DIECI and ELIA (2008).[14]

First, we investigate the implications of randomness. For this purpose, we set $p = q$ which implies a symmetric transition matrix. The chain is then on average half of time in state one and half of the time in state two. We let $p = q$ increase gradually from $1/8$ to $7/8$ in steps of $1/8$. Thereby the mobility of the chain increases according to Shorrocks' index from 0.25 to 1.75. If $p = q = 1/2$, the Markov chain has no memory and the resulting sequence is i.i.d. We are especially interested in the value of ϕ_2^π at which the model switches from being indeterminate to determinate. This will be the case when the minimal Lyapunov exponent λ_{min} crosses the zero line. The corresponding value of ϕ_2^π is denoted by $(\phi_2^\pi)^*$.

Comparing the bifurcation diagrams in Figures 1 and 2, one can see that the stability properties of the model remain qualitatively similar. For low values of ϕ_2^π, the model has two Lyapunov exponents opposite of zero indicating indeterminacy. As the central bank becomes more and more aggressive in combating inflation in state 2, i.e. as ϕ_2^π increases, the two Lyapunov exponents approach each other and the minimal Lyapunov exponent λ_{min} crosses the zero line so that the model becomes determinate. The value at which this happens depends on $p = q$. As the chain becomes more persistent (low values of $p = q$) the aggressiveness of the central bank in state two must increase. When $p = q = 0.25$ which corresponds to a mean exit time of four periods, the value of ϕ_2^π must be at least 3.78 (see also the first part of panel (a) in Table 1) to obtain a determinate model. Note that although the two Lyapunov exponents approach each other as ϕ_2^π increases, they seem to collapse only for high values of $p = q$, i.e. for a highly persistent chain. Note that when $p = q$ becomes low leading to a persistent chain, the line

[14]For further details see FROYLAND et al. (2013) and NEUSSER (2017). In particular, we use 10^7 iterations and a tolerance level of 10^{-6}.

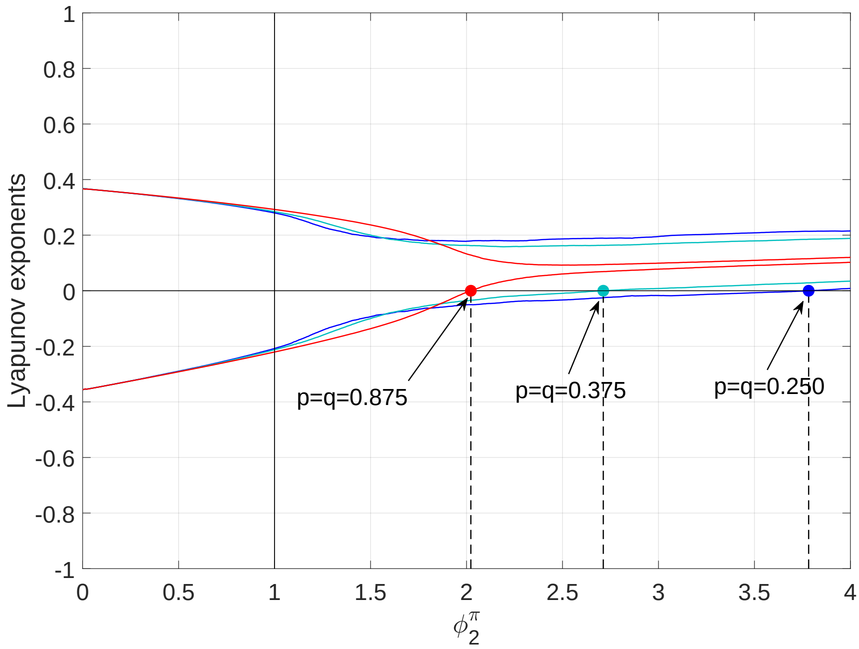

Figure 2: Switching values: The role of Randomness

showing λ_{min} as a function of π_2^π becomes very flat. This implies that the aggressiveness of central bank must become very high and that the precision of the estimate $(\phi_2^\pi)^*$ decreases.

In the next simulation exercise, we fix p at 0.25 and change q gradually in steps of $1/8$ from $1/8$ to $7/8$. This increases the volatility of the chain according to Shorrocks' index. However, in contrast to the previous simulation, the average percentage time spent in state two (active central bank) is thereby successively reduced from 0.666 to 0.222 percent. The details of this specification with the corresponding results are summarized in the second part of panel (a) in Table 1. As expected, the aggressiveness of the central bank must increase strongly to compensated for the lower mean exit time from state two (which is equal to $1/q$). Note that, as before, the precision of the estimate of $(\phi_2^\pi)^*$ tend to decrease as q gets large because λ_{min} viewed as a function of ϕ_2^π becomes very flat.

In a next step, we examine how these results are affected when the central reacts to output in both states. This means that the interest rate is endogenous irrespective in which state the economy is in. The corresponding results for $\phi^y = 1.0$ are presented in panel (b) of Table 1. A comparison with

Table 1: Minimal Lyapunov Exponents: The Role of Randomness

	panel (a): $\phi^y = 0$						
p	0.125	0.250	0.375	0.500	0.625	0.750	0.875
q	0.125	0.250	0.375	0.500	0.625	0.750	0.875
$^{p}/_{p+q}$	0.500	0.500	0.500	0.500	0.500	0.500	0.500
$M(P)$	0.250	0.500	0.750	1.000	1.250	1.500	1.750
$(\phi_2^\pi)^*$	6.12	3.78	2.71	2.27	2.11	2.05	2.02
p	0.250	0.250	0.250	0.250	0.250	0.250	0.250
q	0.125	0.250	0.375	0.500	0.625	0.750	0.875
$^{p}/_{p+q}$	0.666	0.500	0.400	0.333	0.286	0.250	0.222
$M(P)$	0.375	0.500	0.625	0.750	0.875	1.000	1.125
$(\phi_2^\pi)^*$	1.84	3.76	7.09	8.08	8.26	8.10	8.15
	panel (b): $\phi^y = 1$						
p	0.125	0.250	0.375	0.500	0.625	0.750	0.875
q	0.125	0.250	0.375	0.500	0.625	0.750	0.875
$^{p}/_{p+q}$	0.500	0.500	0.500	0.500	0.500	0.500	0.500
$M(P)$	0.250	0.500	0.750	1.000	1.250	1.500	1.750
$(\phi_2^\pi)^*$	1.82	1.82	1.82	1.85	1.87	1.87	1.88
p	0.250	0.250	0.250	0.250	0.250	0.250	0.250
q	0.125	0.250	0.375	0.500	0.625	0.750	0.875
$^{p}/_{p+q}$	0.666	0.500	0.400	0.333	0.286	0.250	0.222
$M(P)$	0.375	0.500	0.625	0.750	0.875	1.000	1.125
$(\phi_2^\pi)^*$	1.37	1.82	2.27	2.74	3.20	3.72	4.21

At $(\phi_2^\pi)^*$ the minimal Lyapunov exponent crosses the zero line.

panel (a) reveals that the aggressiveness of the central bank can now be much lower to achieve a determinate model. In the symmetric case, take f.e. the specification $p = q = 1/2$. Then the value of ϕ^π at which the model switches from being indeterminate to determinate is reduced from 2.27 to 1.85. This reduction is much more dramatic in the asymmetric case. If $p = 1/4$ and $q = 5/8$, the value $(\phi^\pi)^*$ is reduced from 8.26 to 3.20. Thus, the response of the central bank to output has much more effect in the random environment compared to the deterministic one.

4 Conclusion

This paper documented how the Lyapunov exponents can be used to analyze the stability of affine (linear) rational expectations models with time-varying (random) coefficients. In the context of a prototype New Keynesian model the issue of a randomized Taylor rule is analyzed. It is shown how this feature affects the determinateness of the model. In particular, the central bank can compensate periods of a passive policy against inflation by being more aggressive in periods of an active policy. The methods proposed in this paper allow to delineate clearly this trade-off both theoretically as well empirically. Moreover, solution formulas for the determinate as well as the indeterminate case are derived.

The methods outlined in this paper can be readily generalized to analyze models where the randomness of the coefficients are governed by more sophisticated stochastic processes: Markov chains with more than two states or covariance stationary processes. Another interesting generalization relates to the analysis of models with initial conditions. In the context of the New Keynesian model this can be achieved by allowing some inertia in the Phillips curve. The stability of such models could be analyzed in a similar manner. However, a more in depth analysis would require not only to estimate the Lyapunov exponents, but also the Lyapunov spaces. This task is more involved, but numerical algorithms are readily available (Froyland et al., 2013) also for this issue.

As pointed out by Foerster et al. (2016), the approach outlined sofar potentially suffers from two deficiencies. First, the randomness is attached to certain parameters after the linearization of the model. This results in a model which can be seen as being incompatible with the idea that agents take the randomness of certain parameters (policies) already into account in their optimization problem. Against this argument one may object that this view, pushed to the extreme, would make it impossible to investigate any policy changes. Hence, the idea of time-varying coefficients seems to be a reasonable presumption. Second, because of its linearity (i.e. first order approximation), the model fails to adequately represent the effects of time-varying volatility. However, it must be emphasized that the MET can be generalized, at the price of some mathematical sophistication, to nonlinear continuously differentiable random dynamical systems. In this setup and assuming hyperbolicity, Arnold (2003, chapter 7) derived Hartman-Grobman type theorems (Linearization Theorems) which justify the use of linearized systems to infer the qualitative behavior of the original nonlinear system.[15]

[15]For deterministic systems see f.e. Robinson (1999).

Thus, a complete machinery is ready to analyze rational expectations models with random coefficients and thereby to generate new insights.

A Random Dynamical Systems

The purpose of this appendix is to give a precise statement of Oseledets' Multiplicative Ergodic Theorem (MET) which is the theoretical basis for this paper. The presentation draws heavily on Colonius and Kliemann (2014). Other excellent presentations can be found in the monographs by Arnold (2003) and Viana (2014). For a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$, we consider a measurable map $\theta : \mathbb{Z} \times \Omega \to \Omega$ with the properties $\theta(0, \omega) = \mathrm{id}_\Omega$ and $\theta(t+s, \omega) = \theta(t, \theta(s, \omega))$ for all $t, s \in \mathbb{Z}$ and $\omega \in \Omega$. The latter feature is often called the *cocycle* property. θ with these properties is called a (measurable) *dynamical system*. The cocycle property together with two-sided time $\mathbb{Z}$ implies that $\theta(., \omega)$ is invertible. Moreover, as $\theta(t, \omega)$ is generated by $\theta(1, \omega)$, i.e. $\theta(t, \omega) = \theta(1, \omega)^t$, we write $\theta^t\omega$ for $\theta(t, \omega)$ for conciseness. Moreover, we assume that θ is invariant under $\mathbf{P}$, i.e. $\theta(t, F)\mathbf{P} = \mathbf{P}(F)$ for all $F \in \mathfrak{F}$, and that $\mathbf{P}$ is ergodic with respect to θ.

The conditional expectations are defined as $\mathbb{E}_t x_{t+j} = \mathbb{E}[x_{t+j} \mid \mathfrak{F}_t]$, $j \geq 1$, where $\mathfrak{F}_t = \sigma\{(x_s, A_s, b_s) : s \leq t\}$, the smallest σ-algebra such that (x_s, A_s, b_s) is measurable for all $s \leq t$. The sequence of σ-algebras $\{\mathfrak{F}_t\}$ so-defined is a filtration adapted to $\{x_t\}$ and $\{(A_t, b_t)\}$ with $\mathfrak{F}_t \subseteq \mathfrak{F} = \sigma(\bigcup_{t\in\mathbb{Z}} \mathfrak{F}_t)$.

In the context of our simulation exercise randomness is governed by a discrete time finite state regular (ergodic and aperiodic) Markov chain defined by a transition matrix P. Hence, $\Omega = \{1, 2, \ldots, n\}^{\mathbb{Z}}$ where n denotes the number of states. $\theta : \Omega \to \Omega$ is the *shift operator* and $\mathbf{P}$ the associated Markov measure on Ω. As the transition matrix P is irreducible (ergodic), $\mathbf{P}$ is ergodic and invariant with respect to θ. Thus, the assumptions made above are fulfilled.

Consider the nonautonomous linear difference equation with $x_t \in \mathbb{R}^d$ for all $t \in \mathbb{Z}$:

$$x_{t+1} = A(\theta^t\omega)x_t, \quad t \in \mathbb{Z},$$

where $A : \Omega \to \mathbb{GL}(d)$ is measurable and where $\mathbb{GL}(d)$ denotes the general linear group of order d (the set of invertible $d \times d$ matrices). The solutions paths starting with $x_0 = x$ are denoted by $\varphi(t, \omega, x)$ and are given by

$$\varphi(t, \omega, x) = \Phi(t, \omega)x = A(\theta^{t-1}\omega) \ldots A(\omega)x.$$

In the main text, we omit, if possible, the dependence on ω in order not to overload the notation and write A_t for $A(\theta^t\omega)$. The Lyapunov exponents $\lambda(\omega, x)$ are then defined as

$$\lambda(\omega, x) = \limsup_{t\to\infty} \frac{1}{t} \log \|\varphi(t, \omega, x)\|.$$

With these preliminaries we are now in a position to state the MET as in Colonius and Kliemann (2014, section 11.1).

Theorem (Multiplicative Ergodic Theorem (MET))**.** *Let θ be a dynamical system with the properties stated above and assume that the integrability condition*

$$\mathbb{E}\log^+\|A\| \text{ and } \mathbb{E}\log^+\|A^{-1}\| < \infty$$

holds. Then the following assertions follow:

(i) There is a decomposition (splitting)

$$\mathbb{R}^d = L_1(\omega) \oplus \cdots \oplus L_\ell(\omega)$$

of $\mathbb{R}^d$ into $\ell \leq d$ random linear subspaces $L_j(\omega)$. These subspaces are not constant, but depend measurably on ω. However, their dimensions remain constant and equal to d_j. The spaces $L_j(\omega)$ are called Lyapunov or Oseledets spaces.

(ii) The Lyapunov spaces are equivariant, i.e. $A(\omega)L_j(\omega) = L_j(\theta\omega)$.

(iii) There are real numbers $\infty > \lambda_1 > \cdots > \lambda_\ell \geq -\infty$ such that for each $x \in \mathbb{R}^n \setminus \{0\}$ the Lyapunov exponent $\lambda(\omega, x) \in \{\lambda_1, \ldots, \lambda_\ell\}$ exists as a limit and

$$\lambda(\omega, x) = \lim_{t\to\pm\infty} \frac{1}{t} \log \|\varphi(t, \omega, x)\| = \lambda_j \text{ if and only if } x \in L_j(\omega) \setminus \{0\}.$$

(iv) The limit

$$\Upsilon(\omega) = \lim_{t\to\infty} \left(\Phi(t,\omega)'\Phi(t,\omega)\right)^{1/2t} \tag{A.1}$$

exists as a positive definite matrix. The different eigenvalues of $\Upsilon(\omega)$ are constants and can be written as $\exp(\lambda_1) > \cdots > \exp(\lambda_\ell)$; the corresponding random eigenspaces are $L_1(\omega), \ldots, L_\ell(\omega)$.

(v) The Lyapunov exponents are obtained as limits from the singular values δ_k of $\Phi(t, \omega)$: The set of indices $\{1, 2, \ldots, d\}$ can be decomposed into subsets S_j, $j = 1, \ldots, \ell$, such that for all $k \in S_j$,

$$\lambda_j = \lim_{t\to\infty} \frac{1}{t} \log \delta_k(\Phi(t, \omega)).$$

All these assertions hold on some full **P***-measure.*

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